Fracture Analysis in Plane Structures with the Two-scale G/XFEM Method

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Abstract

Generalized or extended finite element method (G/XFEM) uses enrichment functions that hold a priori knowledge about the problem solution. These enrichment functions are mostly limited to two-dimensional problems. A well-established solution for problems without any specific types of analytically derived enrichment functions is to use numerically-build functions in which they are called global-local enrichment functions. These functions are extracted from the solution of boundary value problems defined around the region of interest discretized by a fine mesh. Such solution is used to enrich the global solution space through the partition of unity framework of the G/XFEM. Here it is presented a two-scale/global-local G/XFEM approach to model crack propagation in plane stress/strain and Reissner-Mindlin plate problems. Discontinuous functions along with the asymptotic crack-tip displacement fields are used to represent the crack without explicitly represent its geometries. Under the linear elastic fracture mechanics approach, the stress intensity factor (obtained from a domain-based interaction energy integral) can be used to either determine the crack propagation direction or propagation status, i.e., the crack can start to propagate or not. The proposed approach is presented in detail and validated by solving several linear elastic fracture mechanics problems for both plane stress/strain and Reissner-Mindlin plate cases to demonstrate its the robustness and accuracy.

Keywords: G/XFEM method, Multi-scale fracture analysis, Reissner-Mindlin plate, Shear locking, Crack Growth

1. Introduction

Generalized or extended finite element method (G/XFEM) [4, 13, 48] is a powerful tool to model arbitrary crack geometry and its evolution. This method is defined on the basis of the partition of unity approach [34] that allows to incorporate a priori knowledge about the behavior of the analyzed problem. It also eliminates need for remeshing and conformity to element boundaries. The approximation of G/XFEM is built over a mesh of elements using interpolation functions from the Finite Element Method (FEM). Special functions multiply the original FEM functions and smooth as well as non-smooth solutions can be defined independently of the mesh. Initial implementation of the G/XFEM for crack propagation problems was introduced by Belytschko and Black [4] and Moës et al [38]. After that, this method has been extensively used for the simulation of crack propagation problems, for example [13, 42] for 3D and [49, 52] for 2D problems. A fracture modeling of the Reissner-Mindlin plate was firstly presented by Dolbow et al [10] with the aim of the G/XFEM, in which this work was the first attempt to model the crack propagation in structural problems using the capabilities of the G/XFEM. After that, different researchers have used G/XFEM formulations based on Reissner-Mindlin and Kirchoff plate theories to develop fracture modeling in shell and plate structures, see for example [9, 27, 36, 53]. The Reissner-Mindlin formulation requires only $C^0$ continuity for the approximation, which means that the same partition of unity used for the plane stress/strain problems can be used in this case.

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Another approach that can facilitate the fracture analysis of different problems is to use a two-scale methodology that leads to a more accurate solution with less number of degrees of freedom (DOFs). The generalized/extended finite element method with the global-local enrichment function (G/XFEM) is a combination of two-scale analysis with the G/XFEM method, firstly proposed by Duarte et al [14]. Comparison of the effectiveness of global-local FEM with the G/XFEM was done by Kim et al [25] in which their numerical experiments demonstrate that the G/XFEM is much more robust than the global-local FEM. The global-local strategy based on G/XFEM approach is applied to high-cycle fatigue crack growth in 3D bodies by Pereira et al [43]. A coarse-scale mesh in the G/XFEM doesn’t need to model the crack surface explicitly. Instead, the cracks are modeled through global-local enrichment functions. A two-scale approach using the G/XFEM applied to multi-site cracking problems was presented by Evangelista et al [16] where realistic boundary conditions are applied and multiple cracks with different geometries in a three-dimensional airfield slab are considered. Plews and Duarte [44] used an interdependent solution of global and local problems in order to resolve multi-scale effects due to fine-scale heterogeneities under G/XFEM strategy.

Although there are many investigations on two-scale/multiscale analysis of fracture problems using G/XFEM method, all of them were done only for three-dimensional or plane stress/strain problems, while there is no thorough study for Reissner-Mindlin plate problems. This work aims to cover this lack in the literature by presenting an implementation of the two-scale G/XFEM to model crack propagation in plane structural problems. The focus is the Reissner-Mindlin plate formulation that is used in the two-scale analysis. Discontinuous functions along with the asymptotic crack-tip displacement fields are used to represent the crack without explicitly meshing its surfaces. A domain-based interaction energy integral, based on the J-integral approach [45], is used to extract the stress intensity factor for different fracture modes. Also, maximum circumferential stress criterion is selected for calculation of the crack propagation direction. This work is developed in the INSANE (Interactive Structural Analysis Environment) in-house code. This
computational platform is an open source software available at http://www.insane.dees.ufmg.br. It includes classical, stable and two-scale G/XFEM strategies [1, 28, 30, 31, 32], Meshless method [20], and also quasi-static crack propagation based on G/XFEM approach [33]. An outline of the present paper is as follows. A general formulation of the classical G/XFEM and global-local (two-scale) enrichment is presented in section 2. A brief summary of Reissner-Mindlin plate formulations along with the moment/shear intensity factor calculation procedures are brought in section 3. Section 4 provides a brief explanation along with corresponding formulation for crack propagation process. In section 5, the fracture modeling approach is applied to different linear elastic fracture mechanics problems which emphasizes the main ideas along with robustness and accuracy of these implementations. Beside the fracture analysis in the numerical section, the shear locking phenomenon is investigated for the Reissner-Mindlin plate problem using combination of global-local enrichment function and polynomial enrichments with different orders. Concluding remarks are brought in the final section.

2. Classical and Two-scale Generalized/Extended FEM

2.1. The Generalized/Extended FEM Discretization

The generalized and extended finite element methods are two equivalent approaches (therefore, they can be considered as a single method as G/XFEM), that extrinsically enrich the approximation space of the solution. The G/XFEM was specially developed for modeling structural problems with discontinuities [4, 13, 35]. It is based on the partition of unity method (PUM) [2] that employs a set of partition of unity (PU) functions to guarantee interelement continuity. Such strategy creates conforming approximations which are improved by a nodal enrichment scheme.

The enrichment scheme is obtained by multiplying a PU function of $C^0$ type with compact support (a cloud or patch of elements that share the same nodal point $x_j$) $\omega_j$ by the function $L_{ji}(x)$, named as a local approximation (also called enrichment function). The resulting shape function $\phi_{ji}(x)$ inherits characteristics of both functions, i.e., the compact support and continuity of the PU and the approximate character of the local function. In Fig. 1, in $\mathbb{R}^1$, the PU is obtained from linear Lagrangian functions (represented by $N_i$) associated with each cloud.

As a consequence, the generalized global approximation, denoted by $\tilde{u}(x)$ (decomposed by FE and enriched parts), can be described as a linear combination of the shape functions associated with each node:

$$
\tilde{u}(x) = \sum_{j=1}^{N} N_j(x) u_j + \sum_{j=1}^{N} N_j(x) \left\{ \sum_{i=2}^{q} L_{ji}(x) b_{ji} \right\}
$$

(1)

where $u_j$ is a nodal parameter associated with standard FE shape function - $N_j(x)$, $b_{ji}$ is nodal parameter associated with G/XFEM shape functions - $N_j(x) \cdot L_{ji}(x)$. The terms “standard” and “enriched” refer to the finite element and enriched interpolation fields, respectively, where the standard part is considered as the background field upon which the enriched interpolation field is superimposed. The enrichment function can be either continuous or discontinuous function, depending on the problem type. An example of the enrichment function, $L_{ji}$, by considering the singularities can be defined as [13]:

![Figure 1: Partition of Unity from finite elements for 1D problem](image)
where \( r \) and \( \theta \) are the polar coordinates centered on the crack-tip, and \( \kappa \) is the material constant with \( \kappa = 3 - 4v \) and \( \kappa = \frac{3(1-\nu)}{1+\nu} \) for plane strain and plane stress analysis, respectively. The superscripts \( x \) and \( y \) are referred to \( x \) and \( y \) directions, respectively. These two enrichment functions are obtained from the two-dimensional elasticity solution displacement field in the vicinity of crack-tip [19, 42, 50]. In both expressions, the first terms are the \( x \) and \( y \) components related to the first term of mode-I expansion, respectively. Likewise, the second term of \( x^{L}\text{ja}(x) \) and \( y^{L}\text{ja}(x) \) is extracted from the first term of the mode-II expansion. Another common and useful example of the singular enrichment function in which called the near-tip enrichment, is defined as [4]:

\[
[F_i(r, \theta)] = \left[ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right]
\]

(4)

where \( i \) is the number of crack-tip functions \( F(r, \theta) \) and \( (r, \theta) \) denotes the local polar coordinate defined at the crack-tip. In order to take into account the plate bendings, other types of enrichments must be used, as it was presented in [9] by:

\[
[g_i(r, \theta)] = \left[ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{2} \sin \frac{\theta}{2}, \sqrt{2} \cos \frac{\theta}{2}, \sqrt{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \sqrt{2} \cos \frac{\theta}{2} \cos \frac{\theta}{2} \right]
\]

(5)

In the case of Reissner-Mindlin plate problems, crack-tip functions \( F(r, \theta) \) are used to enrich the degrees of freedom (DOFs) in rotation, and \( g(r, \theta) \) are used to enrich the transverse displacement DOFs. Finally, in a more general case that the crack tip does not coincide with an element edge, the approximation of Eq. (1) for the case of an arbitrary crack takes the form:

\[
\tilde{u}(x) = \sum_{i \in I} N_i(x) u_i + \sum_{i \in I} \bar{N}_i(x) H(x) b_i + \sum_{k \in K} \bar{N}_k(x) \left( \sum_{l=1}^{n} C_{l k} \bar{L}(x) \right)
\]

(6)

where \( J \) is the set of all nodes, \( I \) is the set of nodes enriched with Heaviside function \( H(x) \), and \( K \) is the set of crack-tip nodes. The function \( \bar{L}(x) \) is the crack tip enrichments that can be chosen as the singular enrichment of Eqs. (2) and (3) or near-tip enrichment from (4) and (5), and \( n \) is number of enrichment function need to be used.

2.2. Two-scale G/XFEM

The two-scale/global-local G/XFEM originally proposed by [12], combines the standard G/XFEM with the global-local strategy proposed by [40]. G/XFEM [8] is suitable for problems with local phenomena, such as stress field next to the crack tip. The analysis is divided in three steps: Initial global problem (step 1) that uses a coarse FEM mesh, Local (fine-scale) problem (step 2) which uses a refined mesh in a small part of the initial global problem, and the Final global problem (step 3) that some of the nodes from initial global problem are enriched using numerical functions calculated in step 2. Figure 2 shows the three global-local steps for quasi-static crack propagation at time step \( t \). The time step, \( t \), refers to the number of crack propagation step during the quasi-static analysis.

2.2.1. Initial Global Problem (step 1)

A coarse FEM mesh is used to the whole domain. The position of the cracks can be either between the element edges or inside of the element boundaries. Consider a domain \( \Omega_G = \Omega_G \cup \Gamma_G \) of an elastic problem in \( \mathbb{R}^n \). The boundary is decomposed in \( \Gamma_G = \Gamma_u \cup \Gamma_n \) with \( \Gamma_u \cap \Gamma_n = \emptyset \), where indices \( u \) and \( \sigma \) refer to the Dirichlet and Neumann boundary conditions. \( u^I_G \in X^I_G(\Omega_G) \) represents the solution of the approximate space \( X^I_G(\Omega_G) \) for the initial global problem in its weak form, shown in:
\[ \int_{\Omega_G} \sigma(u_G^t) : \varepsilon(v_G^t) dx = \int_{\Gamma_G} \bar{t} \cdot v_G^t ds \quad (7) \]

where \( \sigma, \varepsilon, v_G^t \in X_G^t(\Omega_G) \), and \( \bar{t} \) are stress tensor, strain tensor, test functions, and prescribed traction vector, respectively.

2.2.2. Fine-scale Problem (step 2)

A refined mesh is used in a small part of the initial global problem. \( \Omega_L^t \) is a sub-domain from \( \Omega_G \). This sub-domain may contain cracks, holes or other special features. The corresponding local solution \( u_L^t \in X_L^t(\Omega_L^t) \) is obtained from:

\[
\int_{\Omega_L^t} \sigma(u_L^t) : \varepsilon(v_L^t) dx + \eta \int_{\Gamma_{L_u} \cap \Gamma_G} u_L^t \cdot v_L^t ds + \kappa \int_{\Gamma_{L^L} \cap \Gamma_G} u_L^t \cdot v_L^t ds = \int_{\Gamma_{L^L} \cap \Gamma_G} \sigma \bar{t} \cdot v_L^t ds + \eta \int_{\Gamma_{L_u} \cap \Gamma_G} \bar{u} \cdot v_L^t ds + \int_{\Gamma_{L^L} \cap \Gamma_G} (t(u_G^t) + \kappa u_L^t) \cdot v_L^t ds \quad (8)
\]

where \( v_L^t \in X_L^t(\Omega_G) \) represents the test functions, \( X_L^t(\Omega_G) \) is the space generated by FEM or G/XFEM functions, \( \eta \) is the penalty parameter and \( \kappa \) is the stiffness parameter to consider Cauchy boundary condition. \( \kappa = 0, \kappa = \eta >> 1, \) and \( 0 < \kappa < \eta \), if Eq. (8) corresponds to a Neumann (if \( \Gamma_{L_u} \cap \Gamma_G^d = \emptyset \)), Dirichlet and Cauchy problem, respectively [26].

Figure 2: Global-local (GL) steps for quasi-static crack propagation. The global solution \( u_G^t \) at a crack propagation time step \( t \) provides boundary conditions for local problem defined in the domain \( \Omega_L^t \) of the crack surface \( \Gamma_L^{t+1} \). Then, the solution of the local problem is used to enrich the global problem at crack propagation time step \( t + 1 \).

2.2.3. Enriched Global Problem (step 3)

Some of the nodes from initial global problem are enriched using numerical functions \( u_L^t \) calculated in step 2. The new solution \( u_G^t \in X_G^t(\Omega_G) \) is obtained from:

\[ \int_{\Omega_G} \sigma(u_G^t) : \varepsilon(v_G^t) dx = \int_{\Gamma_G} \bar{t} \cdot v_G^t ds \quad (9) \]

where \( v_G^t \) represents test functions and \( X_G^t(\Omega_G) \) is the initial space increased by \( u_G^t \) from local problem \( u_L^t \).
\[ X_G' (\Omega_G) = \left\{ \tilde{u}(x) = \sum_{j=1}^{N} N_j(x) \hat{u}_j(x) + \sum_{k \in I_{gl,t}} N_k(x) u_{k_{gl,t}}(x) \right\} \]  

(10)

and \( k \in I_{gl,t} \) represent the set of nodes enriched by the local solution and \( u_{k_{gl,t}}(x) \) is the function obtained from the local solution \( u_{L}^t \) using the Eq. (8).

The global-local cycle shown in Fig. 2 refers to multiple global-local enrichment process during each simulation time step. This means that a complete global-local enrichment strategy is followed for each global-local cycle, i.e., solving global problem, then solving local problem, and finally enriching the global problem. For each crack propagation time step, at least one global-local cycle must be considered. But, to have a better results from the global-local strategy, one can use multiple global-local cycles at the same time step.

The equations presented in this section are for the general 3D problems. However, they can be used for either 2D or Reissner-Mindlin plates. The differences between them are the problem dimension, in other words the degrees of freedom (DOF) corresponding to each node and the interactions between global and local problems. In the case of 2D/plate problem, the DOFs are 2/3 per node, but the boundaries between elements are the same, i.e., a line. However, for the 3D case, not only the DOF are different (3 for solid and 6 for shell problems), but the shared boundaries between elements are surface area, and not line. Therefore, one can modify the 2D implementations by considering additional DOFs for each node, and perform the global-local analysis with small modifications. However, migrating from 2D to 3D or shells deals with more complexities.

3. Reissner-Mindlin Plate

3.1. Formulation

A structural element which is thin and flat is called plate. The thin means that the plate transverse dimension, or thickness, is small compared to the length and width dimensions. The Reissner-Mindlin plate theory are applied to very thin, \( L/t > 100 \), moderately thin, \( 20 < L/t < 100 \), and thick, \( L/t < 20 \), plates, where \( t \) and \( L \) represent the plate thickness and a representative length or width dimension. Reissner-Mindlin plate theory assumes that the normals to the plate do not remain orthogonal to the mid-plane after deformation, thus allowing for transverse shear deformation effects. This allows to use \( C^0 \) approximations.

The assumptions of the Reissner-Mindlin plate theory are the following: (1) \( u = v = 0 \) in the points belonging to the plane \( z = 0 \), which means the points on the middle plane, only move vertically; (2) the points along a normal to the middle plane have the same vertical displacement (i.e., the thickness does not change during deformation); (3) the normal stress \( \sigma_z \) is negligible (plane stress assumption); and (4) straight line normal to the undeformed middle plane remains straight but not necessarily orthogonal to the middle plane after deformation. More importantly, it is assumed that the middle plane is placed exactly at the same distance from the upper and lower faces, see Fig. 3.

Figure 3: A Reissner-Mindlin plate, notations and sign conventions for the displacements and the rotations of the normal.
Assuming the Reissner-Mindlin plate is made of an isotropic homogeneous material, the constitutive relations under bending moment and shear force are given by [10]:

\[
\begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix} = \frac{Et^3}{12(1-\nu^2)}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{bxx} \\
\varepsilon_{byy} \\
\varepsilon_{bxy}
\end{bmatrix}
\] (11a)

\[
\begin{bmatrix}
Q_{xz} \\
Q_{yz}
\end{bmatrix} = \mu kt
\begin{bmatrix}
\varepsilon_{sz} \\
\varepsilon_{sz}
\end{bmatrix}
\] (11b)

where \( M \) and \( Q \) representing the moment and shear force components applied over the plate (see Fig. 3), \( \varepsilon_b \) and \( \varepsilon_s \) are bending and shear deformations, respectively, \( E \) is the Young’s modulus, \( \nu \) is Poisson’s ratio, \( \mu \) is the shear modulus, and \( t \) represents the plate thickness. The correction factor \( k \) accounts for the parabolic variation of the transverse shear stresses through the thickness of the plate, and is taken to be \( k = 5/6 \). For detailed explanation see [10].

3.2. SIF Calculation Procedure

The interaction energy integral method is adapted here to calculate the stress intensity factor, and hence the crack propagation orientation. A summary of the stress intensity factor calculation for Reissner-Mindlin plate is discussed here, while the reader can find detailed formulation in [10]. The J-integral contour for a Reissner-Mindlin plate problem is defined as [10]:

\[
J_x = \int_\Gamma \left\{ W \delta_{x\beta} - [M_{a\beta}(\theta_{a,x} + Q_{\beta}w_x)] n_\beta d\Gamma \right\}
\] (12)

where \( W \) is the strain energy density defined by \( W = 0.5[Ma\beta\theta_{a,x} + Q_{\beta}(\theta_{\beta} + w_x)] \), \( \delta \) is the Kronecker delta, \( Ma\beta \) is the bending moment, \( Q_{\beta} \) is the shear, \( w \) is the transverse displacement and \( \theta_a \) is section rotation about the \( a \) axes that define the middle plane of the plate, in which \( a \) and \( \beta \) ranging over the symbols \( x,y \). In the case of open contour \( \Gamma \) surrounding a crack tip (as shown in Fig. 4), the path independent \( J_x \) integral has a magnitude equivalent to the energy release rate corresponding to a unit crack advance in the \( x \) direction.

![J-integral domain definition with conventions at crack tip](image)

Referring to Fig. 4, one can reach following interaction energy integral equation for the Reissner-Mindlin plate, and for a crack with traction-free faces as:

\[
I = \int_C \left\{ -W \delta_{x\beta} + [M_{a\beta}^{(1)}\theta_{a,1}^{(2)} + M_{a\beta}^{(2)}\theta_{a,1}^{(1)} + Q_{\beta}^{(1)}w_{x,1}^{(2)} + Q_{\beta}^{(2)}w_{x,1}^{(1)}] \right\} m_\beta q dC
\] (13)
in which State (1) represents the current state and State (2) is an auxiliary state. The interaction strain energy, \( W \), is defined by:

\[
W^{(1,2)} = M^{(1)} : \varepsilon_b^{(2)} + Q^{(1)} : \varepsilon_s^{(2)} = M^{(2)} : \varepsilon_b^{(1)} + Q^{(2)} : \varepsilon_s^{(1)}
\]  

(14)

By applying the divergence theorem to the integral over \( C \) (from Fig. 4), one can obtain:

\[
I = \int_A \left\{ \left[ M_{a\beta}^{(1)} \theta_a^{(2)} + M_{a\beta}^{(2)} \theta_a^{(1)} + Q_{a\beta}^{(1)} w_x^{(2)} + Q_{a\beta}^{(2)} w_x^{(1)} \right] - W \delta_{1\beta} \right\} q_\beta \, dA
\]  

(15)

The above integral can be reduced depending on whether the quantity of interest is \( K_I, K_{II}, \) or \( K_{III} \), as certain terms in the auxiliary fields vanish for each case. For \( K_I \) and \( K_{II} \) fracture modes, the integral takes the form:

\[
I = \int_A \left\{ \left[ M_{a\beta}^{(1)} \theta_a^{(2)} + M_{a\beta}^{(2)} \theta_a^{(1)} \right] - W \delta_{1\beta} \right\} q_\beta \, dA
\]  

(16)

whereas for \( K_{III} \) the integral is:

\[
I = \int_A \left\{ \left[ Q_{a\beta}^{(1)} w_x^{(2)} + Q_{a\beta}^{(2)} w_x^{(1)} \right] - W \delta_{1\beta} \right\} q_\beta \, dA
\]  

(17)

The auxiliary state for the displacement fields in Reissner-Mindlin plate theory can be found in Sosa [47] as a power series in \( \sqrt{r} \) and are shown in the Table 3.2.

<table>
<thead>
<tr>
<th>Field</th>
<th>Mode I</th>
<th>Mode II</th>
<th>Mode III</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>( \frac{6 \pi r}{E t^3} \left[ \frac{1}{2} (1 + \nu) C_{\beta} \left( \nu - 1 \right) \right] )</td>
<td>( \frac{6 \pi r}{E t^3} \left[ - \frac{1}{2} (1 + 3 \nu) S_{\beta} - \nu S_{\beta} \right] )</td>
<td>( \frac{6 \pi r}{E t^3} S_{\beta} )</td>
</tr>
<tr>
<td>( \theta_x )</td>
<td>( \frac{6 \pi r}{E t^3} \left[ 4 (1 - \nu) C_{\beta} \right] )</td>
<td>( \frac{6 \pi r}{E t^3} S_{\beta} \left[ 4 (1 + \nu) C_{\beta} \right] )</td>
<td>( \frac{16 \pi r}{E t^3} \left[ - S_{\beta} - (1 + 3 \nu) C_{\beta} S_{\beta} \right] )</td>
</tr>
<tr>
<td>( \theta_y )</td>
<td>( \frac{6 \pi r}{E t^3} \left[ 4 S_{\beta} - (1 + \nu) C_{\beta} S_{\beta} \right] )</td>
<td>( \frac{6 \pi r}{E t^3} \left[ - 2 C_{\beta} - (1 - \nu) S_{\beta} \right] )</td>
<td>( \frac{16 \pi r}{E t^3} C_{\beta} \left[ 1 + (1 + 3 \nu) C_{\beta} \right] )</td>
</tr>
</tbody>
</table>

in which \( C_\theta \) and \( S_\theta \) represent \( \cos \theta \) and \( \sin \theta \) functions, respectively. Moreover, the auxiliary bending moments and shear are as follows:

\[
M_{11} = \frac{K_1}{\sqrt{2r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - \frac{K_2}{\sqrt{2r}} \sin \frac{\theta}{2} \left( 2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)
\]  

(18)

\[
M_{22} = \frac{K_1}{\sqrt{2r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - \frac{K_2}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}
\]  

(19)

\[
M_{12} = \frac{K_1}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_2}{\sqrt{2r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)
\]  

(20)

\[
Q_1 = - \frac{K_3}{\sqrt{2r}} \sin \frac{\theta}{2}, \quad \text{and} \quad Q_2 = \frac{K_3}{\sqrt{2r}} \cos \frac{\theta}{2}
\]  

(21)

As an example, \( K_2 \) and \( K_3 \) must be set equal to zero in all equation in order to calculate auxiliary moment intensity factor of mode-I. The process of evaluating the mixed-mode intensity factors must be carried out with a judicious choice of the auxiliary moment and shear force intensity factors to evaluating the interaction energy integral. From the Eq. (12) and the energy release rate formulation, one can obtain the following expression:

\[
I = \frac{24 \pi}{E t^3} \left[ K_{II}^{(2)} + K_{III}^{(2)} \right] + \frac{12 \pi}{10 \mu t} K_{III}^{(2)} I_{III}^{(2)}
\]  

(22)
where, to extract $K_I$, the following values are chosen: $K_I^{(2)} = 1$, and $K_{II}^{(2)} = K_{III}^{(2)} = 0$. Then, the moment intensity factor $K_I$ can be calculated as:

$$K_I = \frac{E t^3}{24 \pi} I$$  \hspace{1cm} (23)

4. Discontinuity Modeling Procedure

The procedure of discontinuity modeling within a problem in order to simulate the crack propagation is presented here. In the G/XFEM, the discontinuity along with $\Gamma_c$, as shown in Fig. 5, is modeled using the enrichment functions for local part or the whole problem mesh. The signed distance function along with the so-called Heaviside function are used to represent the discontinuity in a model. For linear elastic fracture mechanics, the crack-tip singularity can be captured with the singular enrichment shown in section 2.1 (Eqs. (2) and (3)). The level-set method tracks the motion of an interface by embedding the interface as the zero level-set of the signed distance function. This method is a numerical tool for the tracking of the moving interfaces [46].

Consider a domain $\Omega$ divided into two non-overlapping domains $\Omega_A$ and $\Omega_B$, sharing an interface, or surface of discontinuity, denoted by $\Gamma_c$, as shown in Fig. 5. The signed distance function is defined for the representation of the interface position as:

$$\phi(x) = \| x - x^* \| \cdot \text{sign}(n_{\Gamma_c} \cdot (x - x^*))$$  \hspace{1cm} (24)

where $x^*$ is the closest point projection of $x$ onto the discontinuity $\Gamma_c$, and $n_{\Gamma_c}$ is the vector normal to the interface at point $x^*$. In this definition, $\| \|$ denotes the Euclidean norm, where $\| x - x^* \|$ specifies the distance of point $x$ to the discontinuity $\Gamma_c$ (Fig. 5).

The discontinuity can be represented implicitly as the zero iso-contour of the signed distance function (24) adopted as the level-set function, for which:

$$\phi(x) = \begin{cases} 
  > 0 & \text{if } x \in \Omega_A \\
  = 0 & \text{if } x \in \Gamma_c \\
  < 0 & \text{if } x \in \Omega_B 
\end{cases}$$  \hspace{1cm} (25)

It can be shown that the norm of the gradient of the signed distance level set is equal to unity, that is, $\| \nabla \phi = 1 \|$. Obviously, it is clear that the gradient of the signed distance function at the discontinuity is indeed the unit normal $n_{\Gamma_c}$ oriented to $\Omega_A$, where $\phi(x) > 0$. That is, the $\nabla \phi = n_{\Gamma_c}$ equality holds for the signed distance function at the discontinuity. The discontinuity in the displacement occurs where the displacement of one side of the crack is completely different from the displacement field of the other side. In such cases, the kinematics of the strong discontinuity can be defined based on the Heaviside function [6].
This function is one of the most popular functions used to model the crack discontinuity in the G/XFEM formulation and is defined as:

\[
H(x) = \begin{cases} 
1 & \text{if } \phi(x) > 0 \\
0 & \text{if } \phi(x) < 0 
\end{cases}
\]  

(26)

in which \(\phi(x)\) is the signed distance function, defined in Eq. (24).

5. Numerical Examples

This section presents three linear-elastic problems in two-dimensional domain. Sections 5.1 and 5.2 present a single-edge cracked plate under tension and shear loads, respectively. A cracked Reissner-Mindlin plate is thoroughly analyzed in section 5.3 for a horizontal and inclined crack configuration. In addition to fracture analysis, a shear locking phenomenon is investigated by using different types of enrichment function for Reissner-Mindlin plate to show the characteristics of the present two-scale implementations to overcome this issue for thin plate problems. Linear quadrilateral elements are used to discretize all the examples.

According to Kim et al [26], the Dirichlet boundary condition leads to worse results than Cauchy boundary condition, for transferring BCs to the local boundaries. Thus, this type of BC will be applied on the local problem boundaries in order to demonstrate the robustness of the methodology in the worst case scenario. Numerical integration for the first and second steps of the global-local analysis is done based on standard Gaussian quadrature procedure. In the third step, the numerical integration for those global elements that contain local elements is done over the Gauss points of local elements. Consider that a global element contains \(n'\) local elements and the number of Gauss points for each local element is equal to \(GP\). Thus, the number of integration points for this global element is obtained by: \(\sum_{i=1}^{n'} GP\). The integration order of G/XFEM is for both global and local problems is chosen to be enough to reproduce the polynomial approximation. When there is only a polynomial approximation, the number of integration points is equal to the one necessary to accurately reproduce it. When there is the Heaviside (jump) or singular enrichment functions the number of points must be big enough to minimize the integration error. Therefore, the number of integration points that are used for these problem were selected quite big enough to accurately capture the crack propagation direction within the element domain, specifically for local problem, even for elements containing singular enrichment functions. For simplicity, the same number of integration points is used for the all elements with and without singular/Heaviside enrichment functions. Based on our experience with the modeling approach, the number of global-local cycles for each time step is chosen equal to 3. However, one can select bigger number of global-local cycles, depending on the problem type.

The domain size of the interaction integral is considered here by a circle with radius \(r\) defined by \(r = r_m h_{elem}\) in which the element characteristic length, \(h_{elem}\), is the square root of the crack tip element area and \(r_m\) is a scalar multiplier [38]. To have an accurate SIF results, one have to select a proper multiplier \(r_m\). This scalar multiplier can be chosen by performing numerical experiments with different values to have an independent J-integral path. The crack increment length, \(\Delta a\), should be chosen in such a way to have a reasonable and stable crack propagation procedure. According to [22], an appropriate value must be chosen according to the type of crack propagation, i.e. straight or curved crack, and mesh size to have a reliable crack propagation path. Small values could help to obtain a better accuracy, however, if \(\Delta a\) is too small with respect to the element size, multiple changes in the direction of the crack path may occur which leads to a time consuming element partitioning for numerical integration. The scalar multiplier for all two problems is considered equal to 2.0, but the crack increment length has different values for each problem. The crack propagation criteria and the propagation direction are defined based on the stress field around the crack tip, with the help of the SIF calculation procedure. In addition, the Heaviside (Eq. (26)) and singular (Eqs. (2), (3), (4), and (5)) enrichment functions are used for all problems in order to model the crack lines and capture the crack-tip singularities within the element domains. In all examples, the discontinuity is modeled in the local problem only. Therefore, the first global problem does not include the discontinuity and the third global problem would have the effect of discontinuity by the global-local enrichment function.

The direction of the crack can be determined based on the fracture toughness of brittle material, which is usually measured in a pure mode-I loading conditions by \(K_{IC}\). In this work, the maximum circumferential
tensile stress criterion is used to determine the crack direction angle. This theory was first presented by Erdogan and Sih [15], based on the near the crack-tip state of stress. Based on this theory, the crack propagates perpendicularly to the direction of maximum tension, when \( \sigma_{\theta,max} \) reaches a critical material-dependent constant. In this case, the hoop stress reaches its maximum value on the plane of zero shear stress. The singular term solutions of stress at the crack tip can be used to determine the crack propagation angle, where the shear stress becomes zero. Assuming the mixed-mode loading conditions, the asymptotic crack-tip circumferential stress can be defined in polar coordinate system as [24, 37]:

\[
\sigma_r = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left\{ K_I [1 + \sin^2 \frac{\theta}{2}] + \frac{3}{2} K_{II} \sin \theta - 2 K_{II} \tan \frac{\theta}{2} \right\}
\]

\[
\sigma_\theta = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left\{ K_I \cos^2 \frac{\theta}{2} - \frac{3}{2} K_{II} \sin \theta \right\}
\]

\[
\tau_{r\theta} = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left\{ K_I \sin \theta + K_{II} [3 \cos \theta - 1] \right\}
\]

where \( K_I \) and \( K_{II} \) are stress intensity factors of mode-I and mode-II fracture, respectively, and \( r \) and \( \theta \) are polar coordinate of a point with respect to the crack-tip point. The crack is represented in this work as a set of straight line segments that are connected to each other. It is necessary to compute the critical crack propagation angle, \( \theta_c \), and increment length, \( \Delta \theta \), for the new propagation step. The critical angle can be determined by setting the shear stress \( \tau_{r\theta} \) to zero which leads to:

\[
\frac{\partial \tau_{r\theta}}{\partial \theta} = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left\{ K_I \sin \theta + K_{II} [3 \cos \theta - 1] \right\} = 0
\]

Solving Eq. (28) gives the \( \theta_c \) as follows:

\[
\theta_c = 2 \arctan \left( \frac{1}{4} \left[ \frac{K_I}{K_{II}} \pm \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right] \right)
\]

The result that gives the sign as opposite to sign of \( K_{II} \) is the correct one. Using Eq. (28) in mode-I loading (\( K_{II} = 0 \)), the crack propagation angle is zero. In mode-II loading, by solving the equation \( K_{II} [3 \cos \theta - 1] = 0 \), the crack propagation angle is \( \pm 70.5^\circ \). So the maximum range of the crack propagation angle under linear elastic fracture mechanics approach is limited to an angle range of \([-70.5^\circ \) to \( 70.5^\circ \]). If \( K_{II} > 0 \), the crack growth direction \( \theta_c < 0 \), and if \( K_{II} < 0 \), the crack growth direction \( \theta_c > 0 \). An efficient expression of the critical angle of crack propagation can also be given as:

\[
\theta_c = 2 \arctan \left( \frac{-2K_{II}/K_I}{1 + \sqrt{1 + 8(K_{II}/K_I)^2}} \right)
\]

5.1. Plate with an Edge Crack Under Tension

This example considers a single-edge cracked plate submitted to a tension stress, as shown in Fig. 6. The cracked zone produces singular stress field near the crack tips. The objective of this example is to illustrate the crack propagation under mode-I fracture analysis. The problem is analyzed under plane stress state with (in consistent units): modulus of elasticity \( E = 1.0 \) and Poisson’s ratio \( \nu = 0.3 \).

Figure 6(b) shows the global-local steps and also the local domain discretization along with the global nodes to be enriched with the global-local enrichment function. These three-steps cycle of analysis is repeated for each crack propagation step. There are fourteen global nodes that are enriched with the global-local enrichment function. The singular and the Heaviside enrichment functions are used only in the local mesh.

The problem is discretized with 78 (a regular mesh of \( 6 \times 13 \) elements) and 162 (a regular mesh of \( 18 \times 9 \) elements) elements for global and local models, respectively. The element size of the global problem is \( 1.67 \times 1.54 \), while the local problem has an element size of 0.5. The integration order of the two-scale G/XFEM are \( 8 \times 8 \) and \( 10 \times 10 \) for global and local problem, respectively. In addition, the integration order
Figure 6: (a) Geometry and loading of the single-edge cracked problem, the tension stress is equal to $\sigma = 1.0$. (b) Global-local strategy sequences and local domain (in green) discretization. The black markers indicate the nodes to be enriched with the global-local enrichment function.

for the local problem is chosen big enough to accurately capture the stress intensity factors for each time step, and hence obtain an accurate crack propagation path. The penalty parameter, $\eta$, for Dirichlet boundary condition is chosen equal to $1 \times 10^{12}$. Note that the crack propagation only occurs in the local problem and only its effect are transferred to the global problem via the global-local enrichment function. The crack increment length considered here is equal to 0.325, i.e., almost two-third of the local model element size.

Displacement distributions in $y$ direction along with crack propagation path are shown in Fig. 7 where mode-I crack propagation can be clearly seen from that. This figure belongs to the local problem, where the crack propagation procedure is only active in this second scale. Although there are some small fluctuations in the crack propagation path, but it still remains in the mode-I propagation direction as it was expected.

For the sake of comparison, the crack growth path for a single scale problem with the same mesh as the global mesh from global-local analysis (a mesh with 78 regular elements), with an integration order of $10 \times 10$, are brought in Fig. 8. The same kind of enrichment used in the local problem (Heaviside and singular functions) are employed in this discretization. In contrast to the two-scale problem, the global single scale problem contains the crack and discontinuous enrichments, and therefore the high integration order is used to capture the stress gradient, considering the element size of the single scale problem. As it can be seen from Figs. 7 and 8, the global-local analysis delivers a very good crack propagation path in contrast to the single scale analysis, with a number of DOFs equal to 188, while a quite similar crack propagation path can be obtained with a fine mesh (377 quadrilateral elements with an average size of 0.7) and with 825 DOFs, as it is shown in Fig. 8(c).

In addition, Fig. 9 shows the analytical SIF from Eq. (31) and numerical results, both $K_I$ and $K_{II}$, for the two cases shown in Fig. 8. The maximum error for $K_I$ with respect to analytical values are 11% and 40%, for G/XFEM meshes with the element size ($h$) of 0.7 and 1.5, respectively, and 13% for problem solve with the global-local G/XFEM approach. In addition, the maximum $K_{II}/K_I$ ratio for these three cases are equal to 0.041 (G/XFEM with element size of 0.7), 0.27 (G/XFEM with element size of 1.5) and 0.075 (G/XFEM), which clearly describes the crack path fluctuation for G/XFEM mesh with bigger element size. According to Tada et al [51], the analytical mode-I SIF for problem shown in Fig. 6 is:

$$K_I = [1.12 - 0.231(a/b) + 10.55(a/b)^2 - 21.72(a/b)^3 + 30.39(a/b)^4] \sigma \sqrt{\pi a}$$  (31)
Figure 7: Contour of the displacement in $y$ direction along with the crack propagation path for local problem.

Figure 8: Deformed shape along with the crack propagation path for a single scale problem, for two different element sizes: one with the same as the global problem, and another with smaller element size.
where \( a \) is the crack length, \( b \) is the plate width, and the expression inside of the brackets is an empirical function in which for \( a/b \leq 0.6 \).

Figure 9: Analytical and numerical results of the stress intensity factor for two cases shown in Fig. 8 (both with standard G/XFEM) as well as for Fig. 8 (which is G/XFEM\(^0\)).

5.2. Single-edge Cracked Plate Under Shear Loading

This example corresponds to a rectangular single-edge cracked plate, clamped at the bottom and under the far-field shear stress along the top edge, as it can be seen in Fig. 10(a). Material properties are: Young’s modulus \( E = 3 \times 10^7 \) N/mm\(^2\) and Poisson’s ratio \( \nu = 0.25 \). This problem is analyzed under the plane strain condition and it will be compared with the results from [39] aiming to demonstrate the robustness of the current implementation under the mixed-mode fracture condition.

The problem is discretized with 55 (a regular mesh of 5 \( \times \) 11 elements) and 135 (a regular mesh of 15 \( \times \) 9 elements) elements for global and local models, respectively, as is shown in Fig. 10(b). The element size of the global problem is 1.5, while the local problem has an element size of 0.5. The integration order of the two-scale G/XFEM are as follows: 8 \( \times \) 8 and 10 \( \times \) 10 for global and local problem, respectively. The penalty parameter, \( \eta \), for Dirichlet boundary condition and crack increment length are chosen equal to \( 1 \times 10^{12} \) and 0.325, respectively. Figure 11 shows the crack propagation path for two different propagation steps. These results show a similar pattern to those obtained by Nguyen-Xuan et al [39].
Figure 11: Crack propagation trajectory of plate under shear loading: (a) Step 5 and (b) Step 10 for current implementation obtained from local problem.

5.3. Reissner-Mindlin Plate with a Crack under Bending

A plate is subjected to a far-field moment $M$, as shown in Fig. 12 to have a purely mode-I loading. This problem has the following parameters (in consistent units): modulus of elasticity $E = 1.0$ and Poisson’s ratio $\nu = 0.3$. This problem is divided in three sections: section 5.3.1 provides the validity of intensity factor calculations for the Reissner-Mindlin plate, a shear locking analysis is done in section 5.3.2, a convergence analysis is presented in section 5.3.3, and crack propagation results and explanations are brought in section 5.3.4. In addition, the plate width $W$ is taken to be 20 times the half crack length $a$ in order to have an approximation of the infinite plate, with $a = 0.5$ and $W = 10$. The length of plate is chosen equal to the $W$.

The error in all cases is obtained by: $\text{Error} = \frac{(\text{Solution}_{\text{Numerical}} - \text{Solution}_{\text{Reference}})}{\text{Solution}_{\text{Reference}}}$.

Figure 12: Schematic of the Reissner-Mindlin plate under bending along with the geometry and loading.

Considering the Reissner-Mindlin formulation, the approximation can be of $C^0$ type. As a consequence, it is possible to use the same PoU functions employed for the plane stress and strain problems, i.e., the bi-linear interpolation functions built over the four nodes quadrilateral elements. Those functions are used to approximate, together with the enrichment functions, both the rotations and displacements, described by independent DOF.
5.3.1. Validity of the SIF Calculation

In order to show the capability of the current implementation to extract reasonable moment and shear intensity factors for Reissner-Mindlin plate with an inclined crack, a mesh of 625 quadrilateral elements is used here, see Fig. 13(a). Number of integration points for this analysis is set equal to $8 \times 8 \times 1$. Figure 13(b) shows the extracted moment and shear intensity factor values for mode-I, mode-II and mode-III, covering a full range of the $\beta$ values. As it can be seen from this figure, the results from current work show a good agreement with those obtained from [10].

![Discretization used for moment and shear intensity factor calculation, with \(\beta=0^\circ\) as a schematic.](image)

Figure 13: (a) Discretization used for moment and shear intensity factor calculation, with $\beta = 0^\circ$ as a schematic. $W = 10$, $a = 0.5$, $t_p = 1$, and crack geometry is shown in bold. (b) Normalized moment and shear force intensity factors for the cracked Reissner-Mindlin plate with different crack angle, $\beta$, for current work and from [10].

5.3.2. Shear Locking Study

This section shows the behavior of the two-scale G/XFEM with regard to the plate locking (by changing the plate thickness) considering the degree of the enrichment function as a variable. The locking phenomenon occurs when the approximation space is unable to meet the requirement for allowing null transversal shear deformations as the thickness of the plate goes to zero. Then, the shear deformation energy is overestimated as well as the overall stiffness of the structure. Classical solutions to tackle the shear locking problem in different numerical methods are: selective sub-integration [5, 21] and use of the higher order elements [7] in standard FEM; nodal integration [3] polynomials of different orders to approximate transverse displacements and rotations [11], increasing the polynomial degree of the approximation functions [17]; using mixed formulation [8] in the case of meshless techniques; and using polynomials with different orders for a thick shell element [18] and a square clamped plate [32] for the G/XFEM approach.

![Global mesh along with local domains (in blue).](image)

![Local mesh along with the position of the crack to be used for the locking analysis.](image)

Figure 14: (a) Global mesh along with local domains (in blue), (b) Local mesh along with the position of the crack to be used for the locking analysis.
Figure 14(a) shows the global mesh while the local mesh is shown in Fig. 14(b) in which the black markers (six nodes) are global nodes to be enriched with the global-local enrichment function. Only half of the plate shown in Fig. 12 is modeled due to symmetry along with y direction. The dimension of local domain considered here is equal to 1.90 × 2.25. This domain contains nine global elements with sides of 0.69. The integration order for G/XFEMGL (initial and final global problems as well as local problem) model is considered equal to 8 × 8 × 1. The crack orientation for this analysis is horizontally placed.

Five types of G/XFEMGL analyses are considered here with different combinations of polynomial and numerical enrichment functions. The polynomial enrichment is used to overcome the locking issue for the thin plate case. ‘PG’ and ‘PL’ are the polynomial enrichment order for global and local problems, respectively. The resulting strategies are:

- No polynomial enrichment for both global and local problems (PG = PL = 0).
- No polynomial enrichment for global problem and with first (PG = 0, PL = 1) or second (PG = 0, PL = 2) order for local problem.
- Polynomial enrichment of first (PG = PL = 1) or second (PG = PL = 2) order for both global and local problems.

Figure 15: Normalized moment intensity factor vs. thickness variation for different polynomial orders in global and local problems.

Figure 15 shows the normalized moment intensity factor (with respect to the numerical results obtained from [10]) against the variation of the plate thickness for various enrichment strategies. It can seen from this figure that the results obtained with polynomial enrichments for both global and local problems are closer to the reference solution than the results with polynomial enrichment only for local problem, when the plate thickness goes to smaller values. This means that the polynomial enrichments (for both global and local problems) decrease the effect of shear locking which is in accordance with the conclusions from [17, 18, 32].

Another parameter studied is the error in strain energy for the first and third steps from G/XFEMGL analysis. The reference solution for this part is obtained using a mesh of 34,000 quadrilateral elements (S4, a 4-node doubly curved thin or thick shell quadrilateral element) in ABAQUS®. Figure 16 shows the error in strain energy obtained from first and third steps of the global-local G/XFEM approach. Similar to normalized intensity factor, the results are for using different aforementioned polynomials orders as well as the global-local enrichment function. As it can be seen from Fig. 16, the error gets bigger when the plate thickness approaches smaller values which is due to the locking phenomenon. On the other hand, when polynomial enrichments are used specifically for both global and local problem, the error tends to be less than 25% even for small thickness values.

Both Figures 15 and 16 showed that polynomial and global-local enrichment functions have important impacts to reduce the shear locking effect. Although there were polynomial enrichment and h-refinement in the local problem for the cases of PG = 0, PL = 1 and PG = 0, PL = 2, but since the boundary condition over the local boundaries was poor, both normalized KI and strain energy presented an increase of the error with the reduction of plate thickness. On the other hand, when global problem is enriched with polynomials (either PG = 1 or PG = 2), the accuracy of boundary conditions to be applied over the local domain are
improved, due to effect reduction of the shear locking, and hence the results show better behavior for both normalized $K_I$ and error in strain energy.

5.3.3. Convergence Study

This section presents a convergence study for a Reissner-Mindlin plate problem under bending, as shown in Fig. 17(a). Two different thicknesses are used: 0.1, and 1.0, along with modulus of elasticity of $E = 1.0$ and Poisson’s ratio of $\nu = 0.3$ (in consistent units). There are three different element sizes ($h$) for global mesh in this study, $h = 2.0, 1.0,$ and $0.5$, with total number of elements of 50, 200, and 800, respectively (see Fig. 17(b)).

Figure 17: (a) Geometry of the Reissner-Mindlin plate problem under bending ($M = 1$), and (b) Schemes of three global meshes. Blue elements represent the local domain. The local solution is used to enrich the PU functions associated to the nodes with black markers.

The size of local domain adopted here is equal to $6 \times 8$. This domain is composed by 12, 48, and 192 global elements with $h = 2.0, 1.0,$ and $0.5$, respectively, see colored elements of Fig. 17(b). The local mesh is shown in Fig. 18 and it is fixed with 224 elements for all three global meshes. For the three discretizations, the local problem has exactly the same description with a combination of regular and geometric mesh (with a 10% ratio of element decreasing). Only the four elements from the cloud associated with the crack tip in local problem is discretized with geometric mesh. The reference solution of this problem is obtained using a mesh of 21,200 quadrilateral elements (S8R5, a 8-node reduced integration bilinear quadrilateral shell element with 5 DOFs) in ABAQUS®. The model was restricted to have only displacement in $z$ direction and rotations over $x$ and $y$ directions.
Figure 19 shows the relative error in strain energy against the inverse of element size \((1/h)\) for the two-scale G/XFEM approaches with polynomial of order 0 and 1 (\(PG\) and \(PL\) are polynomial orders of global and local problems, respectively), including both thicknesses. For thickness of 0.1, the G/XFEM\(^{gl}\) with \(PG = PL = 0\) delivers a convergence rate less than 0.5, while a convergence rate of 1.5-1.9 was delivered for polynomials of order 1. On the other hand, for thickness of 1, the convergence rate for \(PG = PL = 0\) and \(PG = PL = 1\) are equal to 1.2 and 1.7-2.0, respectively. The low convergence rate for \(PG = PL = 0\) with thickness of 0.1 was mainly due to effect of shear locking, which was shown in section 5.3.2 that has bigger impact on thinner plates and that can be reduced with polynomial enrichment in the both global and local problems.

5.3.4. Fracture Analysis

This section deals with the fracture analysis of the plate problem shown in Fig. 12. Two different cases are studied here: plate with a horizontal crack, i.e., \(\beta = 0\), and with an inclined crack with \(\beta = 60\) degrees. For horizontal crack, only one-half of the plate is modeled with the finite elements using the symmetry about the \(x_2\) axis. The plate thickness is chosen as \(t_p = 1.0\). The global and local element sizes are equal to 0.7 and 0.25, respectively, with 104 (a regular mesh of \(8 \times 13\) elements) and 216 (a regular mesh of \(24 \times 9\) elements) elements for global and local problems, respectively. The crack increment length considered here is equal to 0.23, in which the crack propagation only occurs in the local problem. The penalty parameter, \(\eta\), for Dirichlet boundary condition is chosen equal to \(1 \times 10^{10}\). Figure 20(b) shows the global as well as the local problem discretization along with the global nodes to be enriched with the global-local enrichment function. There are eighteen global nodes that are enriched with the global-local enrichment function.
Table 2 gives the comparison of extracted moment intensity factors values for mode-I results, showing the error from the exact solution, of the numerical reference from [10], and current G/XFEM and G/XFEMgl analyses. These results are from two steps of the global-local strategy, one obtained in the global problem (first step) and the other in the global-local enriched problem (third step). It can be observed that the solution obtained with the global-local approximation strategy with only 1044 DOFs is quite accurate as the standard G/XFEM approximation with 2868 DOFs. The scalar multiplier \( r_m \) for G/XFEM and G/XFEMgl are equal to 2.5 and 2.0, respectively.

As it was stated before, the number of global-local cycles for each time step was chosen equal to 3. As an example why we have chosen this value, strain energy convergence for the current Reissner-Mindlin plate are brought in Table 3. It can be seen from this table that the strain energy for both steps from the global-local analysis are almost the same after the third cycle, and thus the third cycle seems to produce acceptable results.

Figure 21 presents the rotation distributions over \( x_2 \) direction, i.e., \( \theta_{x_2} \), along with crack propagation path. Similar to section 5.1, the crack propagation procedure is only active in the second scale, i.e., the local problem. A quite pure mode-I of the crack propagation can be clearly seen from this, as it was expected from having a purely mode-I far-field moment.

In the case of inclined crack with \( \beta = 60 \) degrees, the whole plate are modeled with following geometrical parameters: \( a = 0.25 \), \( W = 5 \), and thickness \( t_p = 0.5 \). Again, the \( W/a \) is set equal to 20 to represent an approximation of the infinite plate. The number of global and local elements are equal to 56 and 216, respectively, with an element size of 0.7 for global and 0.25 for the local problem. The crack increment length considered here is equal to 0.153. The integration order for the initial and final global problems as well as local problem are considered equal to \( 8 \times 8 \times 1 \). Figure 22 shows the local problem discretization along with
Figure 21: Contour of the rotation over $x_2$ direction ($\theta_{x_2}$) for Reissner-Mindlin plate, along with the crack propagation path for local problem.

the crack position for plate with the central inclined crack. Similar to the horizontal crack case, there are eighteen global nodes that are enriched with the global-local enrichment function.

Figure 22: Local problem discretization along with the crack position, for Reissner-Mindlin plate with $\beta = 60^\circ$.

Figure 23 presents the crack propagation path for the Reissner-Mindlin plate with an inclined crack. Again, the crack propagation procedure is only active in the second scale, i.e., the local problem. As it can be clearly seen from this figure, the initial inclined crack is propagating horizontally after initial steps of analysis, which is due to the mode-I far-field moment.

6. Final Considerations

The aim of this paper was to present a fracture modeling approach for plane structures (either plane stress/strain or Reissner-Mindlin plate problems) using the capabilities of the two-scale generalized/extended
finite element method. The two-scale approach was implemented in such a way to extend to other types of fine scale models (e.g., meshfree and hp-cloud, among others). The validation of the current work was presented by solving three numerical examples for solid mechanics, aiming to cover all aspects of the current paper, for both plane stress/strain and Reissner-Mindlin plate problems. Heaviside functions along with singular enrichment functions (for both plane stress/strain and Reissner-Mindlin plate problems) were used to facilitate the crack propagation procedure. Also, the global-local enrichment function was used to enrich the global problem employing the precise solution obtained from the local problem. Among different available approaches from the literature, the maximum circumferential tensile stress criterion was used to determine the crack direction angle, which utilizes the SIF values to perform its calculations. The main conclusions from this work can be summarized as:

- The two-scale crack propagation was established in such a way that the crack only propagates in the local problem and its effects were transferred to the global problem via global-local enrichment functions at each time step. Therefore, the next time step will have the effects of the propagated crack in the global problem, and thus they will be transferred to the local problem for the next crack propagation time step. Beside this, for each crack propagation step, multiple global-local cycles need to be applied to obtain a more accurate enrichment for the global problem (3 cycles for this work).

- The shear locking phenomenon is investigated for a Reissner-Mindlin plate problem using G/XFEM method with different enrichment strategies for both global and local problems. The results showed that the polynomial enrichment along with the global-local enrichment function are able to reduce the effect of locking when the thickness approaches to small values, i.e., for thinner plates. Similar behavior can be found for the single scale G/XFEM analysis in [17, 18].

- The convergence analyses have showed that a better convergence rate can be obtained using two-scale approach for Reissner-Mindlin plates with polynomial enrichments, similar to other works using single-scale approaches [17, 18, 32].

Extending the two-scale G/XFEM approach to shell problems is a potential research for future. Another interesting work that can be done based on this research is to apply the two-scale G/XFEM method for
fatigue analysis of plate structures similar to the work of O’Hara et al. [41], using an easy-to-implement approach presented in [29].

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